Prediction of probabilistic settlements via spectral stochastic meshless local Petrov–Galerkin method

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ABSTRACT

This study introduces the prediction of probabilistic settlements with the uncertainty in the spatial variability of Young's modulus to illustrate the preliminary development of a spectral stochastic meshless local Petrov–Galerkin (SSMLPG) method. Generalized polynomial chaos expansions of Young's moduli and a two-dimensional meshfree weak–strong formulation in elasticity are combined to derive the SSMLPG formulation. Because of the local and truly meshless nature, the SSMLPG method is more computationally efficient than available stochastic numerical methods. Two examples further show that SSMLPG-based predictions remain sufficiently accurate even in case of scattered nodes. Therefore, the SSMLPG method can be a valuable alternative for solving stochastic boundary-value problems.

Keywords: Uncertainty Spectral stochastic meshless local Petrov–Galerkin method
Generalized polynomial chaos expansions Meshfree weak–strong formulation

1. Introduction

Many geotechnical problems are ill-defined and affected by a considerable degree of uncertainty. Such uncertainty can result in unreliable predictions for design purposes. For example, the prediction of settlements caused by a loading may be affected by two categories of uncertainties (e.g. [1, 2]):

(a) **Model uncertainty:** This category of uncertainty results from the errors, which are induced by making some assumptions to simplify a model of soil behavior. The infinitesimal strain assumption, which states that the thickness of a soil layer is not significantly reduced by its settlements, denotes such an example. But, this assumption fails in predicting settlements of a soft soil layer, since settlements of a soft soil layer can significantly reduce its thickness.

(b) **Parameter uncertainty:** This category of uncertainty results from the generation of inaccurate variables to predict settlements. These inaccurate variables can be sourced from the gathering of insufficient data of soil properties or inaccurate description of surcharges. The gathering of insufficient data of soil properties may be caused by reasons such as the natural spatial variability of soil properties, acquisition of too few soil samples, and errors associated with the technique used to measure soil properties. The inaccurate description of a surcharge may be caused by the complex scenarios of it. (e.g. [3]).

This study is interested in developing a SSMLPG method for solving a stochastic boundary-value problem including the consideration of parameter uncertainty. This SSMLPG method is chosen to be first applied to predict probabilistic settlements with accounting for the parameter uncertainty contributed by the spatial variability of Young's moduli. A SSMLPG formulation of such a problem is derived by combining the generalized polynomial chaos (GPC) expansion (e.g. [4]) of Young's modulus and a two-dimensional meshfree weak–strong (MWS) formulation in elasticity (e.g. [5]) in which the meshless shape function is constructed by the radial basis function (RBF). Such a MWS formulation includes the application of a local weak form at nodes on or near the natural boundary and a meshfree strong form at other nodes. Thus, the future application of SSMLPG method to solve a stochastic problem including the consideration of uncertainty in boundary geometries will become easier. At least no boundary integrals having uncertain integration routes exist along the essential boundary.

The major motive for developing the SSMLPG method comes from the local and truly meshless nature of the MLPG method [6]. Unlike the spectral stochastic finite element (SSFEM) (e.g. [7]) and the stochastic element-free Galerkin (SEFGM) methods (e.g. [8]), the SSMLPG method will be applied without using finite element meshes or background cells for numerical integration. Therefore, it is more computationally efficient than those existing.
stochastic numerical methods (e.g. [7,8]). Less time is spent to discretize a domain (especially a complex one). Also MLPG results are still enough accurate even in case of scattered nodes [6]. It is expected that the accuracy of SSMLPG results can be also kept, even if nodes are randomly distributed.

The remainder of the succeeding study is organized into six sections. In Section 2, a meshless shape function is equated by the RBF. In Section 3, the derivation of a 2D MWS formulation in elasticity (e.g. [5]) is reviewed. In Section 4, the GPC expansion of Young’s modulus is derived. The resulting expressions in Sections 3 and 4 are combined to derive a SSMLPG formulation in Section 5. In Section 6, the performance of the SSMLPG method is tested. These test results are used to draw some discussion and conclusions in Section 7.

2. Interpolation by radial basis functions

In order to obtain meshless interpolation, a local interpolation or approximation method is required to construct a meshless shape function. Among all available methods (e.g. [9]), the RBF is selected to construct a meshless shape function for solving a stochastic-boundary-value problem.

Suppose \( \Omega \) is a problem domain, \( \mathbf{x} = (x_1, x_2) \) is a vector of spatial coordinates, \( \theta \) is an event in the probability space, and \( u(\mathbf{x}, \theta) \) is a continuous function of \( \mathbf{x} \) and \( \theta \). If \( \Omega_2 \) is an interpolation domain in the neighborhood of \( \mathbf{x}_0 \), the RBF interpolation of \( u(\mathbf{x}, \theta) \) over \( N \) randomly distributed nodes \( \mathbf{x}_i, i = 1, \ldots, N \) within \( \Omega_2 \) is defined by (e.g. [9])

\[
\begin{align*}
  u(\mathbf{x}, \theta) & \approx \sum_{i=1}^{N} N_i(\mathbf{x}_i, \theta) a_i(\mathbf{x}_i, \theta) + \sum_{i=1}^{m} p_i(\mathbf{x}) b_i(\mathbf{x}_i, \theta) = \mathbf{R}^T \mathbf{a} + \mathbf{p}^T \mathbf{b}
\end{align*}
\]

with the constraint

\[
\sum_{i=1}^{N} p_i(\mathbf{x}) a_i(\mathbf{x}_i, \theta) = 0 \quad (j = 1 - m)
\]

where \( R(\mathbf{x}) \) is the RBF, \( a_i(\mathbf{x}_i, \theta) \) and \( b_j(\mathbf{x}_i, \theta) \) denote unknown coefficients to be determined, \( \mathbf{p} = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})]^T \) is a complete monomial basis of order \( m \), \( \mathbf{R} = [R_1(\mathbf{x}), R_2(\mathbf{x}), \ldots, R_N(\mathbf{x})]^T \), \( \mathbf{a} = [a_1(\mathbf{x}_1, \theta), a_2(\mathbf{x}_1, \theta), \ldots, a_N(\mathbf{x}_1, \theta)]^T \), and \( \mathbf{b} = [b_1(\mathbf{x}_1, \theta), b_2(\mathbf{x}_1, \theta), \ldots, b_m(\mathbf{x}_1, \theta)]^T \). In Eq. (1), \( \mathbf{p} = [1, x_1, x_2] \) (\( m = 3 \)) is chosen, since a 2D problem will be subsequently studied. Besides, \( \mathbf{p}^T \mathbf{b} \) denotes a polynomial basis and this term is added for ensuring the pass of standard patch tests and improving the accuracy of numerical results and interpolation stability.

On the other hand, \( R_i(\mathbf{x}) \) is expressed in terms of distances between \( \mathbf{x} \) and \( \mathbf{x}_i \), \( i = 1, \ldots, N \). The multiquadric (MQ) RBF (e.g. [9]) is used to set \( R_i(\mathbf{x}) \) by

\[
R_i(\mathbf{x}) = R_i(\mathbf{r}_i) = \left[ r_i^2 + \left( \varsigma d_i \right)^2 \right]^\frac{1}{2}
\]

where \( || \mathbf{r} || \) denotes the Euclidean distance, \( \varsigma, d_i \geq 0 \) and \( q \) are two shape parameters, and \( d_i \) is a characteristic length that relates to the nodal spacing within \( \Omega_2 \).

The unknown \( a_i(\mathbf{x}_i, \theta) \) and \( b_j(\mathbf{x}_i, \theta) \) values are solved by enforcing Eq. (1) to pass \( \mathbf{x}_i, i = 1, \ldots, N \) within \( \Omega_2 \). Combining the resulting expressions with Eq. (2) gives

\[
\begin{bmatrix}
  \mathbf{R} & \mathbf{P} \\
  \mathbf{P}^T & 0
\end{bmatrix}
\begin{bmatrix}
  \mathbf{a} \\
  \mathbf{b}
\end{bmatrix}
= \begin{bmatrix}
  \mathbf{U} \\
  0
\end{bmatrix}
\]

in which \( r_k = ||x_k - x_0||, k = 1, \ldots, N \). Solving Eq. (4) yields

\[
\begin{bmatrix}
  \mathbf{a} \\
  \mathbf{b}
\end{bmatrix} = \begin{bmatrix}
  \mathbf{R} & \mathbf{P} \\
  \mathbf{P}^T & 0
\end{bmatrix}^{-1} \begin{bmatrix}
  \mathbf{U} \\
  0
\end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix}
  \mathbf{U} \\
  0
\end{bmatrix}
\]

Mathematical studies (e.g. [9]) have concluded that \( \mathbf{A}^{-1} \) is non-singular for scattered nodes; therefore, \( R_i(\mathbf{x}) \), \( i = 1, \ldots, N \) can be applied without causing interpolation singularity problems. After substituting Eq. (6) into Eq. (1), Eq. (1) is further simplified to

\[
\begin{align*}
  u(\mathbf{x}, \theta) &= \mathbf{R}^T \begin{bmatrix}
  \mathbf{p} & \mathbf{p}^T
\end{bmatrix} \begin{bmatrix}
  \mathbf{a} \\
  \mathbf{b}
\end{bmatrix} \\
  &= \mathbf{\Phi}(\mathbf{x}) \mathbf{U} = \sum_{i=1}^{N} \phi_i(\mathbf{x}) u_i(\mathbf{x}, \theta)
\end{align*}
\]

where \( \mathbf{\Phi}(\mathbf{x}) = [\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_N(\mathbf{x})] \) and

\[
\phi_i(\mathbf{x}) = \sum_{i=1}^{N} R_i(\mathbf{r}_i) A_{ik} + \sum_{j=1}^{m} \frac{\partial p_j(\mathbf{x})}{\partial x_k} \bar{A}_{N+j,k} \quad (k = 1 - N, L = 1 - 2)
\]

Since \( R_i(\mathbf{x}), i = 1, \ldots, N \) passes through \( \mathbf{x}_i, i = 1, \ldots, N \), Eq. (8) satisfies the Kronecker delta condition, which is expressed by

\[
\phi_i(\mathbf{x}) = \delta_{ij} \quad \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\]

where \( \delta_{ij} \) is the Kronecker delta.

3. Two-dimensional meshfree weak-strong formulation in elasticity

Suppose isotropic and linearly elastic soil layers and the infinitesimal strain assumption holds. The governing equation of a 2D elastostatic problem has the following tensor form:

\[
\sigma_{ij} + b_i = 0
\]

where \( \sigma_{ij} \) are the stress components corresponding to the displacement field \( u_i \), \( b_i \) denote the body forces, and \( \gamma_{ij} = \partial^2 \mathbf{\gamma}/\partial x_i \partial x_j \). The boundary conditions are defined by

\[
\begin{align*}
  t_i &= \sigma_{ij} n_j \quad \text{on the natural boundary } \Gamma_T \\
  u_i &= \mathbf{u}_i \quad \text{on the essential boundary } \Theta
\end{align*}
\]

where \( t_i \) are the tractions, \( n_j \) are the components of a unit vector \( \mathbf{n} \) outward normal to \( \Gamma \), \( i \), and \( u_i \) are the prescribed tractions and displacements, and \( \Theta = \Gamma_T \cup \Theta \).
3.1. Meshfree strong form for internal nodes and nodes on essential boundaries

Suppose there are \( N_e \) nodes inside \( \Omega \) or on \( \Gamma \). For each node, a local quadrature domain \( \Omega_x \) is selected and \( \Gamma_x \) represents its boundary. If \( \Omega_x \) for a node does not intersect with \( \Gamma_x \), a meshfree strong form of Eq. (11) is applied at this node. Such a form (e.g. [5]) is derived by introducing the stress–strain and strain–displacement relationships to simplify \( \sigma_{ij}, i, j = 1, 2 \) and substituting the resulting expressions into Eq. (11). Considering the plane strain condition and isotropy in material properties, \( \sigma_{ij}, i, j = 1, 2 \) are simplified by

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \begin{bmatrix}
E(1-v) & (1+v)/2 & 0 \\
(1+v)/2 & E(1-v) & 0 \\
0 & 0 & G
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix}
\]

where \( E \) is the Young’s modulus, \( v \) is the Poisson’s ratio, \( \varepsilon_{ij} \), \( i, j = 1, 2 \) denote the strain components. Subtracting Eq. (13) into Eq. (11) leads to

\[
\begin{align}
\frac{E(1-v)}{(1+v)(1-2v)} \left[ \frac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{2(1-2v)} \frac{\partial^2 u_1}{\partial x_2^2} + \frac{1}{2(1-2v)} \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right] + b_1 &= 0 \\
\frac{E(1-v)}{(1+v)(1-2v)} \left[ \frac{\partial^2 u_2}{\partial x_1^2} + \frac{1}{2(1-2v)} \frac{\partial^2 u_1}{\partial x_2^2} + \frac{1}{2(1-2v)} \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right] + b_2 &= 0
\end{align}
\]

(14)

\[
\begin{align}
\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega - \int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Gamma &= 0 \\
\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \omega_2 \, d\Omega - \int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Gamma &= 0
\end{align}
\]

(15)

where \( \beta_i \) is the trial function associated with \( \chi_i, i = 1, \ldots, N_2 \). Since the current study uses the RBF to construct \( \phi \) and \( \phi \) satisfies Eq. (10), neither Lagrangian multipliers nor penalty parameters are used to impose the essential boundary conditions. Simplifying Eq. (15) by the divergence theorem gives

\[
\begin{align}
\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Omega - \int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Gamma &= 0 \\
\int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_2 \, d\Gamma - \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \omega_2 \, d\Omega &= 0
\end{align}
\]

(16)

3.2. Local weak form for nodes on or near natural boundaries

If \( \Omega_x \) for a node interacts with \( \Gamma_x \), a local weak form of Eq. (11) is applied. Deriving this local weak form initiates with the following equation (e.g. [9]):

\[
\int_{\Omega} \left( \sigma_{ij} - \frac{E(1-v)}{(1+v)(1-2v)} \varepsilon_{ij} \right) \omega_1 \, d\Omega = 0
\]

(17)

where \( \beta_i \) is the trial function associated with \( \chi_i, i = 1, \ldots, N_2 \). Since the current study uses the RBF to construct \( \phi \) and \( \phi \) satisfies Eq. (10), neither Lagrangian multipliers nor penalty parameters are used to impose the essential boundary conditions. Simplifying Eq. (15) by the divergence theorem gives

\[
\begin{align}
\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Omega - \int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_1 \, d\Gamma &= 0 \\
\int_{\Gamma} \sigma_{ij} \varepsilon_{ij} \omega_2 \, d\Gamma - \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \omega_2 \, d\Omega &= 0
\end{align}
\]

(16)

3.3. Discrete formulations

Similarly manipulate Eq. (7) to approximate \( u_i, i = 1, 2 \) over \( \Omega_x \) for a node. Substituting the resulting expressions into Eqs. (14) and (20) and summarizing the results into a compact form give

\[
\begin{align}
K_{ii} u_i &= f_i \\
\end{align}
\]

where \( K \) is the global stiffness matrix, \( F \) is the global force matrix, the subscript \( i \) denotes the contribution to \( K \) or \( F \) at \( \chi_i, i = 1, \ldots, N_2 \), and \( u = (u_1, u_2, \ldots, u_{N_2})^T \). \( K \), and \( F_i \) have the components \( (K_{ij}) \) and \( (F_{ij}) \) as follows:

\[
\begin{align}
(K_{ij}) & = \left[ B^T(\Phi) D(\chi_i) B(\Phi) \right]_{ij} \\
(F_{ij}) & = -\left[ B(\Phi) \right]_{ij}
\end{align}
\]

(23a)

\[
\begin{align}
(K_{ij}) & = \int_{\Omega} V_i^T(\Phi) D(\chi_j) B(\Phi) \, d\Omega - \int_{\Gamma} V_i^T(\Phi) D(\chi_j) B(\Phi) \, d\Gamma \\
(F_{ij}) & = \int_{\Omega} W_i(\chi_j) D(\chi_i) B(\Phi) \, d\Omega - \int_{\Gamma} W_i(\chi_j) D(\chi_i) B(\Phi) \, d\Gamma
\end{align}
\]

(23b)

in which the subscripts \( i \) and \( j \) represent the \( i \)th and \( j \)th nodes within \( \Omega_x \) for \( \chi_i, i = 1, \ldots, N_2 \); respectively and

\[
B(\Phi) = \begin{bmatrix}
\frac{\partial \phi}{\partial x_1} \\
\frac{\partial \phi}{\partial x_2} \\
\end{bmatrix}
\]

(24)
Collecting the contributions to $K$ and $F$ at all nodes gives

$$Ku = F$$

Eq. (25) is not ready for use, since the uncertainty contributed by the spatial variability of $E$ has not been resolved. In the next section, the GPC expansion of $E$ is derived. The resulting expansions are subsequently substituted into Eq. (25) to deduce a SSMLPG formulation of Eq. (11).

4. Random field representations by generalized polynomial chaos expansions

A necessary step of developing a stochastic numerical method is representing a continuous-parameter random field by a vector of random variables. Among available choices (e.g. [7]), this study chooses the GPC expression.

The GPC (e.g. [4]) is a generalization of the classical Wiener’s polynomial chaos (PC). This Wiener’s PC is defined as the span of chaos expansions representing a continuous-parameter random field by a vector of random variables. Among available choices (e.g. [7]), this study chooses the GPC expression.

The GPC expansion of $E$ is derived by (e.g. [4])

$$E = \sum_{i=1}^{\infty} a_i(x)\psi_i(\xi) + \sum_{i_1=1}^{\infty} a_{i_1}(x)\psi_1(\xi)$$

$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} a_{i_1i_2}(x)\psi_2(\xi)$$

$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{i_3=1}^{\infty} a_{i_1i_2i_3}(x)\psi_3(\xi) + \cdots$$

which have zero mean and unit variance. For the notational convenience, Eq. (26) is further modified to (e.g. [4])

$$E = \sum_{i=0}^{\infty} \tilde{E}_i \Psi_i(\xi)$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\Psi_i$ represent the multivariate orthogonal polynomial of $\xi$. Usually, $\tilde{E}_0$ and $\tilde{E}_i$ are set to 1 and the mean value of $E_i$ respectively. Meanwhile, there is a one-to-one correspondence between $\Psi_i$ and $\psi_i$ and between $a_0, a_1, a_{i_1}, \ldots$ and $E_i$.

Furthermore, $\Psi_{i_1}, i = 0, \ldots, \infty$ satisfies an orthogonal relationship, $\langle \Psi_i, \Psi_j \rangle = \langle \Psi_i^0 \rangle \delta_{ij} (i, j = 0, \ldots, \infty)$, where $\langle \cdot \rangle$ is the ensemble average. If $f$ and $g$ represent two functions, $\langle f \rangle$ is computed by (e.g. [7])

(a) Continuous case

$$\langle f(\xi), g(\xi) \rangle = \int \cdots \int f(\xi)g(\xi)w(\xi)w(\xi)\cdots w(\xi)\,d\xi_1 \, d\xi_2 \cdots d\xi_n$$

(b) Discrete case

$$\langle f(\xi), g(\xi) \rangle = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} f(\xi)g(\xi)w(\xi)\cdots w(\xi)$$

where $w(\xi_1), w(\xi_2), \ldots w(\xi_n)$ are the weighting functions. Table 1 (e.g. [4]) lists available choices of orthogonal polynomials and corresponding statistical distributions and weighting functions to generate $\Psi_i, i = 0, \ldots, \infty, \xi_1, \xi_2, \ldots, \xi_n$, and $w(\xi_1), w(\xi_2), \ldots w(\xi_n)$ respectively. In addition, an orthogonal relationship $\langle \Psi_i, \Psi_j \rangle = \langle \Psi_i^0 \rangle \delta_{ij} (i, j = 0, \ldots, \infty)$ can be used to get $\tilde{E}_i, i = 0, \ldots, \infty$ as follows:

$$\tilde{E}_i = \frac{\langle E\Psi_i \rangle}{\langle \Psi_i^0 \rangle}$$

In practice, not all $\tilde{E}_i, i = 0, \ldots, \infty$ are computed. For example, only $\tilde{E}_i, i = 0, \ldots, M$ are computed and Eq. (27) is modified to

$$E = \sum_{i=0}^{M} \tilde{E}_i \Psi_i$$

where $M$ is equal to $\frac{P+P}{2} - 1$ (e.g. [4]), $P$ is the highest order of $\Psi$, and $n$ is the total number of uncorrelated random variables.
5. Spectral stochastic meshless local Petrov–Galerkin formulation

Substituting Eq. (30) into Eqs. (23a) and (23b), these two equations are modified to

\[
\begin{aligned}
\mathbf{K}_i &= \sum_{j=0}^{M} \left( \mathbf{B}^T(x_j)\mathbf{D}(x_j)\mathbf{B}(x_j) \right) \Psi_j + \mathbf{M} \left( \mathbf{K}_i \right) \Psi_j \\
\mathbf{F}_i &= \mathbf{R}_i \end{aligned}
\]

(31)

where \( \mathbf{D}(x_j) \) represents the computation of \( \mathbf{D} \) using \( \mathbf{E}_i(x_j) \) \( \mathbf{M} \mathbf{K}_i \) and \( \mathbf{K}_j \), \( i = 1, M \) is the Pochhammer symbol, and \( \beta \) denotes factorial.

4.2 Go to 3.

5. Else if \( Q_5 \) for a node does not interact with \( \Gamma_T (Q_5 \cap \Gamma_T = \emptyset) \) then

5.1 Compute \( \phi_i(x) \), \( k = 1, \ldots, N \) based on \( Q_5 \).

5.2 Compute the contributions to \( \mathbf{K}_i, L = 0, \ldots, M \) and \( \mathbf{F} \) by Eqs. (32) and (23a); respectively.

5.3 Go to 3.

6. Compute \( \mathbf{F}^W_F \), \( k = 0, \ldots, M \).

7. Substitute the results in Steps 4 to 6 into Eq. (35) for solving \( u_j, j = 0, \ldots, M \).

6. Results

To study the performance of Eq. (35), consider a stochastic boundary-value problem [10] in which the settlement behavior of a clay layer located at the YangCheng district, Kaoshing, Taiwan is studied with the uncertainty contributed by the spatial variability of \( E \) is introduced. Fig. 2 illustrates schematically the layout and boundary conditions of this problem, where \( H \) is the thickness of clay layer, \( B \) is the half foundation width, and \( \sigma_0 \) is the magnitude of the foundation loading. As a comparison, a SSFEM package FERUM [11] is also applied to predict settlements. This package includes a module for applying the GPC to represent a random field.

Based on the borehole data [10], two representative types of probabilistic distribution are chosen to describe the spatial variability of \( E \) of the clay layer. The first type of probabilistic distribution is defined by

\[
E = \mathbf{E}[1 + \beta(\mathbf{x})]
\]

(36)

where \( \mathbf{E} \) is the mean value of \( E \) that is independent of \( \mathbf{x} \) and \( \beta(\mathbf{x}) \) is a zero-mean, scalar, homogeneous random field with its autocorrelation function:

\[
\Gamma_T(z) = S^E_\xi \exp \left( -\frac{|z_i|}{\bar{B}^2 + |z_i|/B} \right)
\]

(37)

where \( S^E_\xi \) is the standard deviation of \( E \), \( z = (z_1, z_2) = (x_2 - x_1) \), and \( x_i \) \( i = 1, \ldots, 2 \) are two nodes. Since Eq. (36) suggests the unavailability of analytical solutions of \( \mathbf{u} \), the Monte Carlo simulation (MCS) is generated to understand the accuracy of the SSMPLG-based or SSFEM-based predicted \( \mathbf{u} \). A MCS (e.g. [7]) is implemented by first sampling \( E \) by Eq. (36). Each sample of \( E \) is then substituted into Eq. (25) or FERUM [11] to predict a sample of \( \mathbf{u} \). The mean value \( \mathbf{u} \) and standard deviation \( \mathbf{S}_u \) \( i = 1, \ldots, 2 \) of all the resulting samples of \( \mathbf{u} \) are then computed. If the MCS is performed using a large number of samples of \( E \), the resulting \( \mathbf{u} \) and \( \mathbf{S}_u \) converge to their analytical solutions. Thus, the accuracy of SSMPLG-based or SSFEM-based predictions of \( \mathbf{u} \) can be understood by comparing the SSMPLG-based, SSFEM-based, and MCS-based predictions of \( \mathbf{u} \) and \( \mathbf{S}_u \). This comparison is implemented using the following parameters:

4. Because of symmetry, \( \Omega \) is defined as \( 0 < x_1 < 3B \) and \( 0 < x_2 < H \). Set \( \bar{B} = H = 10 \text{ m} \) and \( \bar{E} = 19.6 \text{ MPa} \).

5. Applying the MQ RBF using \( a_q = 4.0, q = 1.03 \), and \( d_c = 3.0 \) [9].
c. Choose $\Omega_0$ for any point is a circle centered at this point. Set the radius of each $\Omega_0$ equal to 6.25 m. Assume that $\Omega_0$ for a node is a rectangular centered at this node. Set the width and length of each $\Omega_0$ all equal 6.25 m.

d. Represent temporarily $E$ by the Hermite PC. If the accuracy of corresponding predicted $u$ is unsatisfactory, change to another type of the GPC. Choose temporarily $M = 10$ (n = 2 and $P = 3$) and study the accuracy of predicted $u$ versus the decrease of $M$ values subsequently.

e. Choose temporarily $S_e = 0.12E$ in representing $E$. Double this $S_e/E$ value and examine the corresponding accuracy of predicted $u$ subsequently.

f. Generate a meshless discretization of 65 equally spaced nodes. Randomly change the spacing of any two connecting nodes subsequently and examine the accuracy of corresponding predicted $u$. Fig. 3a and b illustrate schematically these two cases of the meshless discretization.

g. Generate one cases of the finite element discretization in executing the FERUM package [11]. Fig. 3c illustrates this finite element discretization.

Meanwhile, the MCS-based predictions of $u_i$ and $S_{ui}$ ($i = 1, \ldots, 2$) are computed by

$$\bar{u}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} u_{ij} \quad \text{and} \quad S_{ui} = \frac{1}{N_i} \sum_{j=1}^{N_i} [u_{ij} - \bar{u}_i]^2$$

where $N_i$ is the total number of samples for implementing the MCS, and the subscript $j$ denotes the $j$th sample of $u_i$. In addition, the SSMLPG-based or SSFEM-based predictions of $u_i$ and $S_{ui}$ ($i = 1, \ldots, 2$) are computed by (e.g. [7])

$$\bar{u}_i = (\bar{u}_i)_j \quad \text{and} \quad S_{ui} = \sqrt{\sum_{j=1}^{M} \bar{u}_j^2 / (M_j)}$$

where $(\bar{u}_i)_j$ is the $j$th component of $\bar{u}_i$, $j = 0, \ldots, M$. Furthermore, the errors between SSMLPG-based or SSFEM-based and MCS-based predictions of $u_i$ and $S_{ui}$ are quantified by the following error estimators:

$$e_{m,i} = \frac{|(u_i)_{MCS} - (u_i)_{SSMLPG}|}{|(u_i)_{MCS}|} \quad \text{and} \quad e_{s,i} = \frac{|(S_{ui})_{MCS} - (S_{ui})_{SSMLPG}|}{|(S_{ui})_{MCS}|}$$

where $\Delta_{m,i}$ and $\Delta_{s,i}$ denote the desired accuracy standard, respectively. If $e_{m,i} < 10\%$ and $e_{s,i} < 10\%$ denote the desired accuracy standard, Fig. 4a and b indicate that the accuracy of SSMLPG-based predicted $u_i$ and $S_{ui}$ is acceptable. As compared to the SSFEM-based predictions of $u_i$ and $S_{ui}$, a better agreement between the SSMLPG-based and MCS-based predictions of $u_i$ and $S_{ui}$ is found, since $e_{m,i}$ and $e_{s,i}$ are less than $\Delta_{m,i}$ and $\Delta_{s,i}$, respectively. In addition, the computation of MCS-based $u_i$ and $S_{ui}$ is found to be time-consuming. Only 2 s is spent to obtain the SSMLPG-based prediction of $u_i$ and $S_{ui}$, while the implementation of a MCS using 5000 samples takes over 840 s.


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**Fig. 2.** Prediction of settlement behaviors of a foundation clay layer [10] ($i = 1, \ldots, 2$) (not to scale).

**Fig. 3.** Nodal distribution for analyzing the benchmark problem (not to scale).

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Using an Intel® Core™ 2 Duo CPU.
10% and 10% are desired, these two figures indicate that this desire can be still accomplished when the SE value is doubled. However, since SE measures the variability of E, Fig. 7a and b also imply that the variability of material properties can be a major reason causing the loss of accuracy of SSMLPG-based predictions of settlements.

Modifying Fig. 4a and b by substituting Fig. 3b for Fig. 3a, Fig. 8a and b show the variation of \( e_{m,i} \) and \( e_{s,i} \) versus \( x_i \) and randomly distributed nodes. These two figures indicate the SSMLPG-based predictions of settlements remain sufficiently accurate, even if the nodes are randomly distributed. The values of \( e_{m,i} \) and \( e_{s,i} \) change slightly, if Figs. 4a and b and 8a and b are compared.

The second type of probabilistic distribution, which is assumed to describe the spatial variability of \( E \), is the uniform distribution. Thus

\[
E = E[1 + \gamma(x)] = E\left[1 + S_E^2(\zeta_1 + \zeta_2)\right]
\]

where \( \gamma(x) = S_E^2(\zeta_1 + \zeta_2) \) represents another zero-mean, scalar, homogeneous random field having the autocorrelation function \( \Gamma_y \), equal to 0 and 0 < \( \zeta_i \) (i = 1, ..., 2) < 1 denote two uncorrelated random numbers.

As shown in Table 1 (e.g. [4]), \( E \) is now represented by the Legendre PC. Similarly manipulating Figs. 4a and b and 5a and b, Figs. 9a and b and 10a and b show the variation of \( e_{m,i} \) and \( e_{s,i} \) versus \( x_i \), different \( N \) values, and \( S_E/E = 0.12 \). Similarly manipulating Figs. 7a and b and 11a and b show the variation of \( e_{m,i} \) and \( e_{s,i} \) versus \( x_i \) and the change of \( S_E/E \) value from 0.1 to 0.2. Modifying Fig. 9a and b by replacing Fig. 3a with Fig. 3b, Fig. 12a and b show the variation of \( e_{m,i} \) and \( e_{s,i} \) versus \( x_i \) and scattered nodes.

Similar to Figs. 4a and b, 5a and b, 9a and b and 10a and b show that the SSMLPG-based predictions of \( S_{m,i} \) and \( S_{s,i} \) (i = 1, ..., 2) approach more closely to the MCS-based predictions of \( u_i \) and \( S_{m,i} \) in addition, Fig. 11a and b still indicate that the increase of \( S_E/E \) can be a major problem causing the loss of accuracy of SSMLPG-based predictions of settlements. Fig. 12a and b show that the SSMLPG-based predictions remain sufficiently accurate, even if the nodes are randomly distributed.
Fig. 7. Variation of $e_{m,i}$ and $e_{s,i}$ ($i = 1, \ldots, 2$) versus $x_i$ and the increase of $\sigma$ values (describing $E$ by Eqs. (36) and (37), $N_s = 5000$, $M = 6$, and using Fig. 3a).

Fig. 8. Variation of $e_{m,i}$ and $e_{s,i}$ ($i = 1, \ldots, 2$) versus $x_i$ ($N_s = 5000$, describing $E$ by Eqs. (36) and (37), $M = 10$, $\sigma = 0.12$, and using Fig. 3b).

Fig. 9. Variation of $e_{m,i}$ and $e_{s,i}$ ($i = 1, \ldots, 2$) versus $x_i$, different $N_s$ values, and $\sigma = 0.1 (E = \bar{E} + S_2 (\sigma_1 + \sigma_2), M = 10$, and using Fig. 3a).

Fig. 10. Variation of $\Delta_{m,i}$ and $\Delta_{s,i}$ ($i = 1, \ldots, 2$) versus $x_i$ and different $N_s$ values, and $\sigma = 0.1 (E = \bar{E} + S_2 (\sigma_1 + \sigma_2), M = 10$, and using Fig. 3a).
7. Discussion and conclusion

This study illustrates the preliminary development of the SSMLPG method through the prediction of probabilistic settlements with the uncertainty contributed by the spatial variability of $E$. Section 6 has shown that the SSMLPG method is more computationally efficient than the existing SSFEM and SEFGM methods [7,8]. Applying this SSMLPG method needs only scattered nodes. However, a better agreement between the SSMLPG-based and MCS-based predictions can be obtained, as compared to the SSFEM-based predictions. In reality, Section 6 implies that the SSMLPG method can directly incorporate the soil properties measured at discrete points. But the settlements predicted by this new stochastic numerical method are more accurate than the ones provided by existing stochastic numerical methods, even when the uncertainty in the spatial variability of a material property is considered.

In addition, Section 6 also shows that the accuracy of SSMLPG-based predictions reduces, if the $SE/E$ value is doubled. In a practical problem, it is therefore suggested that the $SE/E$ value should be kept as low as possible for obtaining sufficiently accurate SSMLPG-based predictions. Since $SE$ measures the variability of $E$, a possible way of obtaining sufficiently low $SE/E$ values is carefully defining the size of each $X_Q$ such that the variability of $E$ or the corresponding $SE/E$ value is limited. Since the SSMLPG method inherits the local nature of the MLPG method [6], each $X_Q$ can be arbitrary and re-sizing it to limit the variability of $E$ is not difficult.

In conclusion, the SSMLPG method provides more computational efficiency than existing stochastic numerical methods. As compared to the SSFEM-based predictions, a better agreement between the SSMLPG-based and MCS-based predictions is obtained. Hence, the SSMLPG method can be a valuable alternative for solving stochastic boundary-value problems.

References